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Translated by M. D. F.

ON THE UNLOADING WAVE IN MATERIALS WITH DELAYED YIELDING

PMM Vol. 32, №6, 1968, pp. 1097-1099
Iu. P. GULIAEV and V. S. LENSKII
(Moscow)
(Received April 4, 1968)

The question of the existence of an unloading wave in the case of one-dimensional wave propagation in a semi-infinite rod of material with delayed yielding is solved herein. Unloading conditions are formulated here, and an analytic method to obtain an expression for the initial velocity of the unloading wave is expounded.

1. Unloading condition. The dependence $\sigma \sim \varepsilon \sim t$ is taken in the form $\sigma = F(\varepsilon, t - x/a_0)$ in [1] for problems of active longitudinal wave propagation in a rod from material with delayed yielding. In particular, the solution is investigated for the law

$$\sigma = E\varepsilon, \quad |\varepsilon| \leqslant \varepsilon_s, \quad \sigma = E_1\varepsilon + (E - E_1)\,\varepsilon_s(t - x/a_0), \quad |\varepsilon| > \varepsilon_s \tag{1.1}$$

which corresponds to linear hardening upon instantaneous loading. Here ε_s is a monotonely decreasing function of its argument. Henceforth considering only the tension case $(\sigma \geqslant 0,\ \varepsilon \geqslant 0)$, let us note that the requirement for a growth in stress in the cross section, in particular, of loading on the endface of a semi-infinite rod is not necessary. Indeed, defining the plastic deformation as $\varepsilon^p = \varepsilon - \sigma / E$, as is customary, we see that the transition to passive strain is determined by the requirement

$$E\frac{\partial\varepsilon}{\partial t} - \frac{\partial\sigma}{\partial t} \leqslant 0 \tag{1.2}$$

By using (1.1) this condition can be represented in one of the two forms

$$\frac{\partial\varepsilon}{\partial t} \leqslant \varepsilon_s'\left(t - \frac{x}{a_0}\right) \quad \text{or} \quad \frac{\partial\sigma}{\partial t} \leqslant E\varepsilon_s'\left(t - \frac{x}{a_0}\right) \tag{1.3}$$

The limiting case of "neutral" loading investigated in [1] (domain 2 in Fig. 1, and Formula (13) in the mentioned paper) corresponds to the equality sign in (1.3). In particular, unloading at the end of the rod starts at time $t = t_0$ if the applied stress $\varphi(t) = \sigma(0, t)$ satisfies the condition

$$\varphi'(t) \leqslant E\varepsilon_s'(t) \qquad \text{for} \quad t \geqslant t_0 \tag{1.4}$$

2. Unloading wave. Let τ be the time of origination of plastic deformation at the end $x = 0$ of a semi-infinite rod, and let condition (1.4) be satisfied from the time $t = t_0 \geqslant \tau$. Let us show that the boundary between the active and passive strain domains is $t = f(x)$ in the plane of the characteristics (x, t), i.e. the unloading wave has a finite propagation velocity $b = 1/f'(x)$ satisfying the condition

$$a_1 \leqslant b \leqslant a_0, \qquad a_0 = \sqrt{E/\rho}, \qquad a_1 = \sqrt{E_1/\rho} \tag{2.1}$$

Here a_0, a_1 are propagation velocities of the longitudinal elastic and plastic waves, respectively. For the unloading domain we take the connection between the stress and strain as $\sigma - \sigma_0 = E(\varepsilon - \varepsilon_0)$, or according to (1.1)

$$\sigma = E\varepsilon - (E - E_1)(\varepsilon_0 - \varepsilon_{S0}) \tag{2.2}$$

Here $\varepsilon_0(x)$ is the maximum value of the strain at the section x; ε_{S0} is the value of $\varepsilon_S(t - x/a_0)$ at $t = f(x)$. This hypothesis corresponds to observations, in particular, to "freezing" of relaxation, and the time of passage of active strain during unloading, being manifested in the exhaustion of the delay period by repeated impacts [2 and 3]. The equation of motion in the unloading domain $t > f(x)$, if it exists, will be according to (2.2)

$$\frac{\partial^2 u}{\partial t^2} = a_0^2 \frac{\partial^2 u}{\partial x^2} + (a_0^2 - a_1^2)\frac{d}{dx}(\varepsilon_{S0} - \varepsilon_0)$$

and its general solution will be

$$u(x, t) = F_1(x + a_0 t) + F_2(x - a_0 t) - \frac{a_0^2 - a_1^2}{a_0^2}\int_0^x (\varepsilon_{S0} - \varepsilon_0)\, dx \tag{2.3}$$

Let us recall [1] that in the active strain domain

$$u = -\frac{a_1}{E_1}\int_\tau^{t-x/a_1} [\varphi(\xi) - E\varepsilon_S(\xi)]\, d\xi - a_0\int_\tau^{t-x/a_0} \varepsilon_S(\xi)\, d\xi - \frac{a_0}{E}\int_0^\tau \varphi(\xi)\, d\xi \tag{2.4}$$

$$\varepsilon = \frac{1}{E_1}\Big[\varphi(t - x/a_1) - E\varepsilon_S(t - x/a_1)\Big] + \varepsilon_s(t - x/a_1)$$

If the desired unloading wave is a strong discontinuity wave, then from the kinematic and dynamic conditions $b[u_x] = -[u_t] \quad \rho b[u_t] = -[\sigma] \equiv -E[u_x]$

it follows that $b = a_0$ since the jumps in the mentioned quantities are nonzero. Therefore, the unloading wave exists for a jump reduction in the loading at the end of the rod, and it propagates at the elastic wave velocity.

Let us investigate the case of a weak-discontinuity unloading wave when

$$u_x = \varepsilon_0, \quad u_t = u_{t0} \quad \text{for} \quad t = f(x) \tag{2.5}$$

where, according to (2.5), we understand u_t, to be the expression

$$u_{t0} = -\frac{a_1}{E_1}\left[\varphi\left(f(x) - \frac{x}{a_1}\right) - E\varepsilon_S\left(f(x) - \frac{x}{a_1}\right)\right] - a_0\varepsilon_S\left(f(x) - \frac{x}{a_0}\right) \tag{2.6}$$

Computing u_x and u_t on the basis of (2.3), and utilizing (2.4)–(2.6), we find

$$F_1'(x + a_0 f(x)) = \frac{a_1(a_0 - a_1)}{2a_0^2}(\varepsilon_{S0} - \varepsilon_0) \quad F_2'(x - a_0 f(x)) = \frac{a_1(a_0 + a_1)}{2a_0^2}(\varepsilon_0 - \varepsilon_{S0}) + \varepsilon_{S0}$$

Using these expressions and computing the quantity $\partial\varepsilon / \partial t$ on the unloading wave by means of (2.3), we obtain

$$\frac{\partial \varepsilon}{\partial t} = \frac{a_1^2 - b^2}{E_1(a_0^2 - b^2)}\left[\varphi'\left(f(x) - \frac{x}{a_1}\right) - F\varepsilon_S'\left(f(x) - \frac{x}{a_1}\right)\right] + \varepsilon_S'\left(f(x) - \frac{x}{a_0}\right) \tag{2.7}$$

If the line $t = f(x)$ is actually an unloading wave, then according to the first inequality of (1.3), the following inequality should hold:

$$\frac{a_1^2 - b^2}{E_1(a_0^2 - b^2)}\left[\varphi'\left(f(x) - \frac{x}{a_1}\right) - E\varepsilon_S'\left(f(x) - \frac{x}{a_1}\right)\right] \leqslant 0 \tag{2.8}$$

Let us investigate it in order to clarify the properties of the unloading wave, i.e. the conditions under which it exists.

The velocity of the unloading wave cannot exceed the velocity of the elastic wave. Indeed, if $b > a_0$ (meaning $b > a_1$), then it is clear geometrically that $f(x) - x/a_1 < t_0$. But the expression in square brackets in (2.8) is hence nonnegative, and the inequality (2.8) is not satisfied.

If we put $b < a_1$ (meaning $b < a_0$) then the unloading wave will not intersect the active strain domain, which contradicts its definition.

Therefore, the inequality (2.8) is valid under the conditions $a_1 \leqslant b \leqslant a_0$, if the loading at the end of the rod is subject to the requirement (1.4).

3. Initial velocity of the unloading wave. On the basis of the results in Sects. 1 and 2 let us obtain a formula for the initial velocity of the unloading wave. Let us first assume that the pressure $\varphi(t)$ changes its velocity by a jump at the initial unloading point on the plane of characteristics ($t = t_0$, $x = 0$) so that the following relationships hold:

$$k_1 = \varphi'(t_0 - 0), \qquad k_2 = \varphi'(t_0 + 0)$$

Evidently k_1 and k_2 should satisfy the inequalities

$$k_1 \geqslant E\varepsilon_S'(t_0), \qquad k_2 < E\varepsilon_S'(t_0)$$

On the basis of (2.4) and (2.7), the connection between the ultimate strain rates at the rod endface ($x = 0$) is given by Formula

$$\frac{\partial \varepsilon^+}{\partial t} = \frac{a_1^2 - b^2}{a_0^2 - b^2} \frac{\partial \varepsilon^-}{\partial t} + \frac{a_0^2 - a_1^2}{a_0^2 - b} \varepsilon_S'(t_0) \tag{3.1}$$

But according to the laws of linear hardening and unloading, we have at this same point

$$\frac{\partial \varepsilon^+}{\partial t} = \frac{k_2}{\rho a_0^2}, \qquad \frac{\partial \varepsilon^-}{\partial t} = \frac{k_1}{\rho a_1^2} + \frac{a_1^2 - a_0^2}{a_1^2} \varepsilon_S'(t_0) \tag{3.2}$$

Substituting (3.2) into (3.1) we obtain an algebraic equation for the initial velocity of the unloading wave

$$\frac{k_2}{\rho a_0^2} = \frac{k_1 (a_1^2 - b^2)}{\rho a_1^2 (a_0^2 - b^2)} + \frac{b^2 (a_0^2 - a_1^2)}{a_1^2 (a_0^2 - b^2)} \varepsilon_S'(t_0) \tag{3.3}$$

from which it results that

$$b(0) = \left(\frac{a_1^2 a_0^2 (k_1 - k_2)}{a_0^2 k_1 - a_1^2 k_2 - \rho a_0^2 (a_0^2 - a_1^2) \varepsilon_S'(t_0)} \right)^{1/2} \tag{3.4}$$

In particular, if $k_2 = -\infty$, i.e. the pressure at the end of the rod diminishes abruptly, then $b \equiv a_0$.

If $k_1 = k_2 = E\varepsilon_s'(t_0)$, then (3.4) becomes meaningless. The relationship (3.1) is hence satisfied identically. In order to find $b(0)$ in the smooth unloading case, it is sufficient to assume continuity of $\varphi''(t)$ at the initial unloading point.

The process of determining $b(0)$ reduces to the following. Let us calculate the limit values of the second derivatives of the strain with respect to time at $x = 0$, $t = t_0$

$$\frac{\partial^2 \varepsilon^+}{\partial t^2} = \frac{a_1}{2} \left[\frac{a_0 + a_1}{(1 - a_0 f')^2} - \frac{a_0 - a_1}{(1 + a_0 f')^2} \right] \left[\frac{d^2 \varepsilon_0}{dx^2} - \varepsilon_S' f'' - \frac{(a_0 f' - 1)^2}{a_0^2} \varepsilon_S'' \right] + \varepsilon_S''$$

$$\frac{\partial^2 \varepsilon^-}{\partial t^2} = \frac{a_1^2}{(a_1 f' - 1)^2} \frac{d^2 \varepsilon_0}{dx^2} - \frac{a_1^2 (a_0 f' - 1)^2}{a_0^2 (a_1 f' - 1)^2} \varepsilon_S'' - \frac{a_1^2}{(a_1 f' - 1)^2} f'' \varepsilon_S'$$

It hence follows that

$$\frac{\partial^2 \varepsilon^+}{\partial t^2} - a(f') \frac{\partial^2 \varepsilon^-}{\partial t^2} = \varepsilon_s''(t_0) [1 - a(f')] \tag{3.5}$$

$$a(f') = \frac{1}{2a_1} \left[\frac{a_0 + a_1}{(1 - a_0 f')^2} - \frac{a_0 - a_1}{(1 + a_0 f')^2} \right] (a_1 f' - 1)^2$$

On the other hand, the linear hardening and unloading laws yield

$$\frac{\partial^2 \varepsilon^+}{\partial t^2} - \frac{a_1^2}{a_0^2} \frac{\partial^2 \varepsilon^-}{\partial t^2} = \varepsilon_S''(t_0) \left(1 - \frac{a_1^2}{a_0^2}\right) \tag{3.6}$$

The relationship (3.6) will be a consequence of the continuity of $\varphi''(t)$ at $t = t_0$. Equating (3.5) and (3.6), we obtain an algebraic equation to determine the initial velocity of the unloading wave

$$a_1 b^2 + 2a_0^2 b - 3a_0^2 a_1 = 0 \tag{3.7}$$

Solving (3.7) we find

$$b(0) = a_0 \left[\left(\frac{a_0^2}{a_1^2} + 3\right)^{1/2} - \frac{a_0}{a_1}\right] \tag{3.8}$$

Formula (3.8) holds even in the absence of the delayed yielding effect, and is presented in [4].

The expressions obtained for the initial velocity of the unloading wave will be the starting point for constructing all unloading waves by the method of characteristics.

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STABILITY OF STEADY HELICAL MOTIONS OF A RIGID BODY IN A FLUID

PMM Vol. 32, №6, 1968, pp. 1100-1105

N. S. TSODOKOVA
(Moscow)

(Received February 28, 1968)

The problem of existence and stability of steady helical motions of a rigid body bounded by a simply connected syrface, was studied by Liapunov [1], using the Routh's theorem and its complement. Steklov in [2] established the existence of steady helical motions of a rigid body bounded by a multiply connected surface. Below we investigate the stability of the motions found by Steklov using the Routh's theorem and the Liapunov's complement, and we obtain the necessary conditions as well as some sufficient conditions of stability.

1. Let us suppose that a rigid body with several cavities filled with a perfect fluid, moves in an infinite, homogeneous, incompressible perfect fluid. We assume that the space occupied by the fluid (bounded by the surface of the body) and the cavities, are all multiply connected. We also assume that no forces act on the body and the fluid and that the motion of the fluid is irrotational. Taking any three mutually perpendicular straight lines rigidly connected to the body as the $OXYZ$-coordinate system, we shall denote the projections of the velocity of the origin on these axes by u, v and w and by p, q and r the projections of the angular velocity of the body. The principal rotations